ABSTRACT

Ranking systems are a fundamental ingredient of multi-agent environments and Internet Technologies. These settings can be viewed as social choice settings with two distinguished properties: the set of agents and the set of alternatives coincide, and the agents’ preferences are dichotomous, and therefore classical impossibility results do not apply. In this paper we initiate the study of incentives in ranking systems, where agents act in order to maximize their position in the ranking, rather than to obtain a correct outcome. We consider several basic properties of ranking systems, and fully characterize the conditions under which incentive compatible ranking systems exist, demonstrating that in general no such system satisfying all the properties exists.

Categories and Subject Descriptors

G.2.2 [Discrete Mathematics]: Graph Theory; J.4 [Social and Behavioral Sciences]: Economics; H.3.3 [Information Storage and Retrieval]: Information Search and Retrieval

General Terms

Algorithms, Economics, Human Factors, Theory

Keywords

Ranking systems, multi-agent systems, incentives, social choice

1. INTRODUCTION

The ranking of agents based on other agents’ input is fundamental to multi-agent systems (see e.g. [14]). Moreover, it has become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google’s PageRank algorithm[11] and eBay’s reputation system[13].

The ranking systems setting can be viewed as a variation of the classical theory of social choice[4], where the set of agents and the set of alternatives coincide. Specifically, we consider dichotomous ranking systems, in which the agents vote for a subset of the rest of the agents. This is a natural representation of the web page ranking setting[17], where the Internet pages are represented by the agents/alternatives, and the links are represented by votes.

Although the above mentioned work consists of a significant body of rigorous research on ranking systems, the study did not consider the effects of the agents’ incentives on ranking systems1. The issue of incentives has been extensively studied in the classical social choice literature. The Gibbard–Satterthwaite theorem [9, 15] shows that in the classical social welfare setting, it is impossible to aggregate the rankings in a strategy-proof fashion under some basic conditions. The incentives of the candidates themselves were considered in the context of elections[8], where a related impossibility result is presented. Another notion of incentives was considered in the case where a single agent may create duplicates of itself[7]. Furthermore, the computation of equilibria in the more abstract context of ranking games was also discussed[6].

In this paper we initiate research on the issue of incentives in ranking systems. We define two notions of incentive compatibility, where the agent is concerned with its expected position in the ranking under affine or general utility functions.

We then consider some very basic properties of ranking systems, which are satisfied by almost all known ranking systems, and prove that these properties cannot be all satisfied by an incentive compatible ranking system. This finding is far from trivial, as different ranking systems may require different manipulations by an agent in order to increase its rank in different situations. Furthermore, we show that when we assume only a subset of the basic properties, some artificial incentive compatible ranking systems can be constructed. Together, these results form a complete characterization of incentive compatible ranking systems under these conditions.

1A recent work on quantifying incentive compatibility of ranking systems[5] was based on a preliminary version of this paper.
2. RANKING SYSTEMS

Before describing our results regarding ranking systems, we must first formally define what we mean by the words “ranking system” in terms of graphs and linear orderings:

**Definition 1.** Let $A$ be some set. A relation $R \subseteq A \times A$ is called an ordering on $A$ if it is reflexive, transitive, and complete. Let $L(A)$ denote the set of orderings on $A$.

**Notation 1.** Let $\preceq$ be an ordering, then $\succeq$ is the equality predicate of $\preceq$, and $\prec$ is the strict order induced by $\preceq$. Formally, $a \succeq b$ if and only if $a \preceq b$ and $b \preceq a$; and $a \prec b$ if and only if $a \preceq b$ but not $b \preceq a$.

Given the above we can define what a ranking system is:

**Definition 2.** Let $G_V$ be the set of all directed graphs on a vertex set $V$ that do not include self edges. A ranking system $F$ is a function that for every finite vertex set $V$ maps graphs $G \in G_V$ to an ordering $\preceq_F \in L(V)$.

One can view this setting as a variation/extension of the classical theory of social choice as modeled by [4]. The ranking systems setting differs in two main properties. First, in this setting we assume that the set of voters and the set of alternatives coincide, and second, we allow agents only two levels of preference over the alternatives, as opposed to Arrow’s setting where agents could rank alternatives arbitrarily.

3. BASIC PROPERTIES OF RANKING SYSTEMS

In order to classify the incentive compatibility features of ranking systems, we must first define the criteria for the classification. We define some very basic properties that are satisfied by almost all known ranking systems. Most properties have two versions – one weak and one strong, both satisfied by almost all known ranking systems.

First of all, we define the notion of a trivial ranking system, which ranks any two vertices the same way in all graphs.

**Definition 3.** A ranking system $F$ is called trivial if for all vertices $v_1, v_2$ and for all graphs $G, G'$ which include these vertices: $v_1 \leq_{F} v_2 \Leftrightarrow v_1 \leq_{F'} v_2$. A ranking system $F$ is called nontrivial if it is not trivial.

A ranking system $F$ is called infinitely nontrivial if there exist vertices $v_1, v_2$ such that for all $N \in \mathbb{N}$ there exists $n > N$ and graphs $G = (V, E)$ and $G' = (V', E')$ s.t. $|V| = |V'| = n$, $v_1 \leq_F v_2$, but $v_2 \leq_{F'} v_1$.

A basic requirement from a ranking system is that when there are no votes in the system, all agents must be ranked equally. We call this requirement minimal fairness.

**Definition 4.** A ranking system $F$ is minimally fair if for every graph $G = (V, \emptyset)$ with no edges, and for every $v_1, v_2 \in V$: $v_1 \preceq_F v_2$.

Another basic requirement from a ranking system is that as agents gain additional votes, their rank must improve, or at least not worsen. Surprisingly, this vague notion can be formalized in (at least) two distinct ways: the monotonicity property considers the situation where one agent has a superset of the votes another has in the same graph, while the positive response property considers the addition of a vote for an agent between graphs. This distinction is important because, as we will see, the two properties are neither equivalent, nor imply each other.

**Notation 2.** Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex. The predecessor set of $v$ is $P_G(v) = \{v' | (v', v) \in E\}$. The successor set of $v$ is $S_G(v) = \{v' | (v, v') \in E\}$. We may omit the subscript $G$ when it is understood from context.

**Definition 5.** Let $F$ be a ranking system. $F$ satisfies weak positive response for all graphs $G = (V, E)$ and for all $(v_1, v_2) \in (V \times V) \setminus E$, and for all $v_3 \in V \setminus \{v_1, v_2\}$: Let $G' = (V, E \cup \{v_1, v_2\})$. Then, $v_1 \leq_F v_2$ implies $v_1 \leq_{F'} v_2$ and $v_3 \leq_{F'} v_2$ implies $v_3 \leq_F v_2$. $F$ furthermore satisfies strong positive response if $v_1 \leq_F v_2$ implies $v_3 \leq_F v_2$.

**Definition 6.** A ranking system $F$ satisfies weak monotonicity if for all $G = (V, E)$ and for all $v_1, v_2 \in V$: If $P(v_1) \subseteq P(v_2)$ then $v_1 \leq_F v_2$. $F$ furthermore satisfies strong monotonicity if $P(v_1) \subseteq P(v_2)$ additionally implies $v_1 \leq_F v_2$.

**Example 1.** Consider the graphs $G_1$ and $G_2$ in Figure 1. Further assume a ranking system $F$ ranks a $\preceq_{G_1} d$ in graph $G_1$. Then, if $F$ satisfies weak positive response, it must also rank a $\preceq_{G_2} d$ in $G_2$. If $F$ satisfies the strong positive response, then it must strictly rank a $\preceq_{G_2} d$ in $G_2$. However, if we do not assume a $\preceq_{G_1} d$, $F$ may rank a and d arbitrarily in $G_2$.

Now consider the graph $G_1$, and note that $P(a) = \{c\} \subseteq \{c, d\} = P(b)$. This is the requirement of the weak (and strong) monotonicity property, and thus any ranking system $F$ that satisfies weak monotonicity must rank a $\preceq_{G_1} b$, and it is strong monotonicity, it must strictly rank a $\preceq_{G_1} b$.

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Our results are still correct when allowing self-edges, but for the simplicity of the exposition we assume none exist.

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A stronger notion of fairness, the isomorphism property, will be considered in Section 8.
Note that the weak monotonicity property implies minimal fairness. This is due to the fact that when no votes are cast, all vertices have exactly the same predecessor sets and thus must be ranked equally.

Yet another simple requirement from a ranking system is that it does not behave arbitrarily differently when two sets of agents with their respective votes are considered one set.

**Definition 7.** Let $F$ be a ranking system and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs s.t. $V_1 \cap V_2 = \emptyset$ and let $v_1, v_2 \in V_1 \cup V_2$. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. $F$ satisfies the weak union condition if $v_1 \preceq_{G_1 \cup G_2} v_2 \Leftrightarrow v_1 \preceq_{G_1} v_2 \lor v_1 \preceq_{G_2} v_2$. Let $G' = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where $E \subseteq V_1 \times V_2$ is in an arbitrary set of edges from $V_1$ to $V_2$. $F$ satisfies the strong union condition if $v_1 \preceq_{G_1} v_2 \Leftrightarrow v_1 \preceq_{G'} v_2$.

Surprisingly, we will see that even the weak union condition has great significance towards the existence of a ranking system or lack thereof. One reason for this effect, is that a ranking system satisfying this condition cannot behave differently depending on the size of the graph.

### 3.1 Satisfiability

As we have mentioned above, these properties are very basic and, with the exception of the strong union condition, all the properties above are satisfied by almost all known ranking systems such as the PageRank[11] ranking system (with a damping factor) and the authority ranking by the Hubs&Authorities algorithm[10]. These ranking systems do not satisfy the strong union condition, as in both systems outgoing links outside an agent’s strongly connected component may affect ranks inside the strongly connected component, either by dividing the importance (in PageRank) or by affecting the hubbiness score in Hubs&Authorities.

Furthermore, the simple approval voting ranking system satisfies all the strong properties mentioned above including the strong union condition. The approval voting ranking system can be defined as follows:

**Definition 8.** The approval voting ranking system $AV$ is the ranking system defined by:

$$v_1 \preceq_{AV} v_2 \Leftrightarrow |P(v_1)| \leq |P(v_2)|.$$

**Fact 1.** The approval voting ranking system $AV$ satisfies minimal fairness, strong monotonicity, strong positive response, the strong union condition, and infinite nontriviality.

These facts lead us to believe that the properties defined above (perhaps with the exception of the strong union condition), should all be satisfied by any reasonable ranking system, at least in their weak form. We will soon show that this is not possible when requiring incentive compatibility.

### 4. INCENTIVE COMPATIBILITY

Ranking systems do not exist in empty space. The results given by ranking systems frequently have implications for the agents being ranked, which are the same agents that are involved in the ranking. Therefore, the incentives of these agents should in many cases be taken into consideration.

In our approach, we require that our ranking system will not rank agents better for stating untrue preferences, but we assume that the agents are interested only in their own ranking (and not, say, in the ranking of those they prefer).

We assume that for strict rankings (with no ties), for every agent count $n$, there exists a utility function $u_n : N \to \mathbb{R}$ that maps an agent’s rank (i.e. the number of agents ranked below it) to a utility value for being ranked that way. We assume $u_n$ is nondecreasing, that is every agent weakly prefers to be ranked higher.

This utility function can be extended to the case of ties, by treating these as a uniform randomization over the matching strict orders. Thus the utility of an agent with $k$ agents strictly below it and $m$ agents tied is

$$E[u_n] = u_n'(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i).$$

We can now define the utility of a ranking for an agent as follows:

**Definition 9.** The utility $u^*_G(v)$ of a vertex $v$ in graph $G = (V, E)$ under the ranking system $F$ and utility function $u$ is defined as

$$u^*_G(v) = u^*\left([\{v' : v' \prec v\}, \{v' : v' \equiv v\}\right] = \frac{1}{|\{v' : v' \equiv v\}|} \sum_{i=[v' \prec v]} u_n(i).$$

This definition allows us to define a preference relation over rankings for each agent. Using this preference relation, we can now define the general notion of incentive compatibility as immunity of utility to manipulation of outgoing edges:

**Definition 10.** Let $F$ be a ranking system. $F$ is called incentive compatible under utility function $u$ if for all graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ s.t. for some $v \in V$, and for all $v' \in V \setminus \{v\}$, $v'' \in V : (v', v'') \in E_1 \Leftrightarrow (v', v'') \in E_2$: $u^*_{G_1}(v) = u^*_{G_2}(v)$.

A strong notion of incentive compatibility is compatibility under any utility function:

**Definition 11.** Let $F$ be a ranking system. $F$ satisfies strong incentive compatibility if for any nondecreasing utility function $u : N \times N \to \mathbb{R}$, $F$ is incentive compatible under $u$.

A simple utility function one may consider is the identity function $u_a(k) \equiv k$. This basic utility function means that any change in rank has the same significance. The utility of
a ranking with $k$ weaker agents and $m$ equal agents under this function:

$$u_n(k, m) = \frac{1}{m} \sum_{i=k}^{m-1} u_n(i) = k + \frac{m - 1}{2}.$$ 

It turns out that the preference relation over rankings produced by the identity utility function is the same as the one produced by any affine utility function $u(k) = a \cdot k + b$, as $u_n(k, m)$ in this case is simply $a \cdot (k + \frac{m - 1}{2}) + b$. Therefore, it is interesting to look at incentive compatibility under an affine utility function $u$:

**Definition 12.** Let $F$ be a ranking system and let $F$ is called weakly incentive compatible if for every utility function $u : N \times N \rightarrow \mathbb{R}$ such that $u_n(k) = a \cdot k + b$ for some constants $a, b \in \mathbb{R}$, $F$ is incentive compatible under $u$.

**Notation 3.** In order to prevent ambiguity, in the remainder of this paper we will use $r^F_n(v)$ ("rank") to denote $u^F_n(v)$ under the utility function $u_n(k) = k + \frac{1}{2}$. So that

$$r^F_n(v) = \left| \{ v' : v' < v \} \right| + \frac{1}{2} \left| \{ v' : v' \simeq v \} \right| .$$

Note that due to the fact that all affine ranking functions give the same ordering over $u^F_n(k, m)$, we can, wlog, consider only $u_n(k) = k + \frac{1}{2}$ when proving weak incentive compatibility or lack thereof.

Interestingly, we will see in the remainder of this paper that these incentive compatibility properties are very hard to satisfy, and no common nontrivial ranking system satisfies them. In particular, the PageRank, Hubs&Authorities, and Approval Voting ranking systems mentioned above are not weakly incentive compatible.

**Example 2.** One may think that under positive response, impossibility of weak incentive compatibility is a direct result of an alleged dominant strategy not to vote for any agent.

However, this is not true, as sometimes the best response does involve voting for some agent. Consider the ranking system $F$ defined by:

$$v_1 \leq^F v_2 \iff |P(v_1)| + \frac{1}{2} |S(v_1)| \leq |P(v_2)| + \frac{1}{2} |S(v_2)| .$$

This ranking system satisfies strong positive response, but is not weakly incentive compatible. For example, in the graph depicted in Figure 2, the agent $a$ can improve its rank either by not voting for $b$, or by voting for both $x_1$ and $x_2$. The maximal increase in $a$’s rank is achieved by doing both.

Note that under this ranking system, agents do not have a dominant strategy that maximizes their rank, and thus there is no general dominant deviation that demonstrates lack of incentive compatibility.

**5. POSSIBILITY WITHOUT MINIMAL FAIRNESS**

To begin our classification of the existence of incentive compatible ranking systems, we first consider ranking systems which do not satisfy minimal fairness. We have already seen that minimal fairness is implied by weak monotonicity, so we cannot hope to find weak monotonicity without minimal fairness. As it turns out, the strong versions of all the remaining properties considered above can, in fact, be satisfied simultaneously.

**Proposition 1.** There exists a ranking system $F_1$ that satisfies strong incentive compatibility, strong positive response, infinite nontriviality, and the strong union condition.

**Proof.** Assume a lexicographic order $<$ over vertex names, and assume three consecutive vertices $v_1 < v_2 < v_3$. Then, $F_1$ is defined as follows (let $G = (V, E)$ be some graph):

$$v \leq^F_1 v' \iff [v < u \land (v < v_2 \lor u < v_3)] \lor [v = v_2 \land u = v_3 \land (v_1, v_2) \notin E] \lor [v = v_3 \land u = v_2 \land (v_1, v_2) \in E] .$$

That is, vertices are ranked strictly according to their lexicographic order, except when $(v_1, v_2) \in E$, whereas the ranking of $(v_2, v_3)$ is reversed.

$F_1$ satisfies strong positive response because the ordering of the vertices remains unchanged by anything but the $(v_1, v_2)$ edge, and is always strict. The addition of the $(v_1, v_2)$ edge only increases the relative rank of $v_2$ as required.

Assume for contradiction that $F_1$ does not satisfy the strong union condition. Then, there exist two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and an edge set $E \subseteq V_1 \times V_2$ such that the ranking $\leq^F_1$ of graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$ is inconsistent with $\leq^F_1$. First note that the only inconsistency that may arise is with the ranking of $v_2$ compared to $v_3$. Therefore, $(v_2, v_3) \not\subseteq V_1$. Furthermore, for the ranking to be inconsistent $(v_1, v_2) \notin E_1$ and $(v_1, v_2) \in E_2 \cup E$ (the opposite is impossible due to inclusion). Furthermore, $v_2 \in V_1 \Rightarrow v_2 \notin V_2 \Rightarrow (v_1, v_2) \notin V_1 \times V_2 \Rightarrow (v_1, v_2) \notin E$. Thus we conclude that $(v_1, v_2) \in E_2$, and thus $v_2 \in V_2$, in contradiction to the fact that $v_2 \in V_1$. \qed

**6. FULL CLASSIFICATION UNDER MINIMAL FAIRNESS**

We are now ready to state our main results:

**Theorem 1.** There exist weakly incentive compatible, infinitely nontrivial, minimally fair ranking systems $F_2, F_3, F_4$ that satisfy weak monotonicity; weak positive response; and the weak union condition respectively. However, there is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies any two of those three properties.

**Theorem 2.** There is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies either...
one the four properties: strong monotonicity, strong positive response, the strong union condition and strong incentive compatibility.

The proof of these two theorems is split into ten different cases that must be considered – three possibility proofs for $F_2$, $F_3$, and $F_4$, three impossibility results with pairs of weak properties, and four impossibility results with each of the strong properties. We will now prove each of these cases.

### 6.1 Possibility Proofs

**Proposition 2.** There exists a weakly incentive compatible ranking system $F_2$ that satisfies minimal fairness, weak positive response, and infinite nontriviality.

**Proof.** Let $v_1, v_2, v_3$ be some vertices and let $G = (V, E)$ be some graph, then $F_2$ is defined as follows:

$$v \preceq u \Leftrightarrow (v \neq v_3 \land u \neq v_2) \lor v = u \lor (v_1, v_2) \notin E \lor v_2 \notin V.$$

That is, $F_2$ ranks all vertices equally, except when the edge $(v_1, v_3)$ exists. Then, $F_2$ ranks $v_2 < v \simeq u < v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

$F_2$ satisfies minimal fairness because when no edges exist, the clause $(v_1, v_3) \notin E$ always matches, and thus all vertices are ranked equally, as required. $F_2$ satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices $v_1, v_2, v_3$ and the edge $(v_1, v_3)$, which is ranked nontrivially.

$F_2$ satisfies weak positive response because the only edge addition that changes the ranks of the vertices in the graph (the addition of $(v_1, v_3)$) indeed doesn’t weaken the target vertex $v_3$.

$F_2$ is weakly incentive compatible because only $v_1$ can affect the ranking of the vertices in the graph (by voting for $v_3$ or not), but $r(v_1)$ is always $\frac{|V|}{2}$.

**Proposition 3.** There exists a weakly incentive compatible ranking system $F_3$ that satisfies minimal fairness, the weak union condition, and infinite nontriviality.

**Proof.** Let $v_1, v_2, v_3$ be some vertices and let $G = (V, E)$ be some graph, then $F_3$ is defined as follows:

$$v \preceq u \Leftrightarrow (v \neq v_3 \land u \neq v_2) \lor v = u \lor \{(v_1, v_2), (v_1, v_3)\} \notin E.$$ 

That is, $F_3$ ranks all vertices equally, except when the edges $(v_1, v_2), (v_1, v_3)$ exist. Then, $F_3$ ranks $v_2 < v \simeq u < v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

$F_3$ satisfies minimal fairness because when no edges exist, the clause $\{(v_1, v_2), (v_1, v_3)\} \notin E$ always matches, as required. $F_3$ satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices $v_1, v_2, v_3$ and the edges $\{(v_1, v_2), (v_1, v_3)\}$, which is ranked nontrivially.

To prove $F_3$ satisfies the weak union condition, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be some graphs such that $V_1 \cup V_2 = \emptyset$, and let $G = G_1 \cup G_2$. If $\{(v_1, v_2), (v_1, v_3)\} \notin E_1 \cup E_2$ then by the definition of $F_3$, it must rank all vertices in all graphs $G_1, G_2, G$ equally, as required. Otherwise, for all $v, u \in (V_1 \cup V_2) \setminus \{v_2, v_3\}$: $v \preceq_{G_3} u \preceq_{G_3} v_3$. Assume wlog that $(v_1, v_2) \in E_1$ and thus $v_1 \in V_1$. But then also $(v_1, v_3) \in E_1$ and thus also $v_3 \in V_1$. By the definition of $F_3$, for all $v, u \in V_1 \setminus \{v_2, v_3\}$: $v \preceq_{G_3} u \preceq_{G_3} v_3$. As $v_1, v_2, v_3 \notin G_2$, trivially for all $v, u \in V_2$: $v \preceq_{G_3} u$, as required.

$F_3$ is weakly incentive compatible because only $v_1$ can affect the ranking of the vertices in the graph (by voting for $v_3$ or not), but $r(v_1)$ is always $\frac{|V|}{2}$.

**Proposition 4.** There exists a weakly incentive compatible ranking system $F_4$ that satisfies minimal fairness, weak monotonicity, and infinite nontriviality.

**Proof.** The ranking system $F_4$ ranks all vertices equally, except for graphs $G = (V, E)$ for which $|V| \geq 7$, where $V = \{w, s, m_0, \ldots, m_{n-1}\}$, and for all $i \in \{0, \ldots, n-1\}$: $(m_i, s) \in E$, $(m_i, w) \notin E$, and for all $j \in \{0, \ldots, n-1\}$: $(m_i, m_j) \in E$ if and only if $j = (i + 2)$ mod $n$. Figure 3 includes an example graph that satisfies these conditions. In such graphs, $F_4$ ranks $w \preceq_{G_0} m_1 \preceq_{G_0} \ldots \preceq_{G_0} m_n \preceq_{G_0} s$. $F_4$ is minimally far by definition, as when there are no edges, all vertices are ranked equally. $F_4$ satisfies infinite nontriviality because such nontrivially ranked graphs $G$ exist for all $|V| \geq 7$.

$F_4$ satisfies weak monotonicity because in the graphs that it doesn’t rank all vertices equally we see that $P(w) \not\supseteq P(m_i) \not\supseteq P(s)$ for all $i \in \{0, \ldots, n-1\}$, which is consistent with the ordering $F_4$ specifies.

To prove $F_4$ is weakly incentive compatible, we let $G_1, G_2$ be two graphs that differ only in the outgoing edges of a single vertex $v$, and show that $r_{G_2}^F(v) = r_{G_1}^F(v)$. Because all graphs in which not all vertices are ranked equally are of the form defined above, at least one of the graphs $G_1, G_2$ must have this form. Let us assume wlog that this graph is $G_1$, and mark the vertices of this graph as defined above.

Now consider two cases:
1. If $v = w$ or $v = s$, then by the definition of $F_4$, $F_4 = F_3$, so trivially, $r_{F_3}(v) = r_{F_4}(v)$, as required.

2. If $v = m_i$, for some $i \in \{0, 1, \ldots, n - 1\}$, then first note that $r_{F_3}(v) = \frac{1}{|v|}$. If $G_2$ is not of the form defined above then all its vertices are ranked equally and specifically $r_{F_2}(v) = \frac{1}{|v|}$ as required. Otherwise, $G_2$ is of the form defined above. Let $w'$ and $s'$ be the $w$ and $s$ vertices for $G_2$ in the form defined above. By the definition, $2 \leq |P_{G_2}(w')| \leq 4$, while $|P_{G_2}(s')| \leq 1$ and $|P_{G_2}(s')| \geq 5$. Therefore, $v \notin \{w', s'\}$. By the definition of $F_4$, $r_{F_4}(v) = \frac{1}{|v|}$, as required.

6.2 Impossibility proofs with pairs of weak properties

We prove the impossibility results with pairs of weak properties, by assuming existence of a ranking system and analyzing the minimal graph in which the ranking system does not rank all agents equally. This is done in the following lemma:

**Lemma 1.** Let $F$ be a weakly incentive compatible minimally fair nontrivial ranking system. Then, there exists a graph $G = (V, E)$ and vertices $v_1, v_2, v \in V$ such that:

1. For all graphs $G' = (V', E')$ where $|E'| < |E|$ or $|E'| = |E|$ and $|V'| < |V|$, $v_1 \preceq_{G'} v_2$ for all $v_1, v_2 \in V'$.

2. $r_{F}(v) = \frac{1}{|v|}$

3. $v_1 \preceq_{F} v \preceq_{F} v_2$

4. For all $v' \in V$: $v_1 \preceq_{F} v' \preceq_{F} v_2$.

5. $S(v) \neq \emptyset$ and for all $v' \in V$ such that $S(v') \neq \emptyset$: $v' \preceq_{F} v$.

**Proof.** Let $G = (V, E)$ be an arbitrary (edges, then vertices) graph such that there exist $v_1, v_2, v \in V$ satisfying condition 1. Let $v_1, v_2$ be vertices such that for all $v' \in V$: $v_1 \preceq_{F} v' \preceq_{F} v_2$ (such vertices exist because $F$ is nontrivial). Note that these vertices satisfy condition 4.

6.3 Impossibility proofs with the strong properties

**Proposition 8.** There exists no weakly incentive compatible minimally fair ranking system that satisfies the weak monotonicity and weak positive response conditions.

**Proof.** Assume for contradiction a ranking system $F$ that satisfies the conditions. First note that $F$ is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_1, v_2 \in V$ be the graph and the vertices from the lemma.

Now, let $(v_1, v_2) \in E$ be some edge. Let $G' = (V, E \setminus \{(v_1, v_2)\})$. By condition 1, $v_2 \preceq_{G'} v_1$. Let weak positive response, $v_T \preceq_{G'} v_2$. Since this is true for all $v_2 \in V$ with $P(v_2) = \emptyset$, and $v_1 \preceq_{G'} v_2$, we conclude that $P_G(v_1) = P_G(v) = \emptyset$. Now, by weak monotonicity $v_1 \preceq_{G} v$, in contradiction to the fact that $v_1 \preceq_{G'} v_2$.

**Proposition 9.** There exists no weakly incentive compatible nontrivial ranking system that satisfies the strong monotonicity and weak union conditions.

**Proof.** Assume for contradiction a ranking system $F$ that satisfies the conditions. First note that $F$ is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_1, v_2 \in V$ be the graph and the vertices from the lemma.

Now, let $G' = (V \cup \{x\}, E)$ be a graph with an additional vertex $x \notin V$. By the weak union condition, $v_1 \preceq_{F} v_2$. By weak monotonicity, $x \preceq_{F} v_1$. Therefore, by the weak union condition, $r_{F_G}(x) = r_{F}(v) + 1 = \frac{1}{|v|} + 1$. Let $G'' = (V \cup \{x\}, E \setminus \{(x', v') \in V\})$. By condition 1 and the fact that $S_G(v) \neq \emptyset$, $r_{F_G}(v) = \frac{1}{|v|}$. From weak monotonicity, $r_{F_G}(v) = r_{F}(v)$, which is a contradiction.

**Proposition 10.** There exists no weakly incentive compatible nontrivial minimally fair ranking system that satisfies the weak union and weak positive response conditions.

**Proof.** Assume for contradiction a ranking system $F$ that satisfies the conditions. As the conditions of Lemma 1 are satisfied, let $G = (V, E)$ and $v, v_1, v_2 \in V$ be the graph and the vertices from the lemma. Now let $G_1 = (V \setminus \{v_1\}, E)$ and let $G_2 = (\{v_1\}, \emptyset)$. From conditions 3 and 5, $S_G(v_1) = \emptyset$. If $P_G(v_1) \neq \emptyset$, then by condition 1 in the graph $G'' = (V, E \setminus \{(x, v_1)\})$ where $x \in F_G(v_2), v_2 \preceq_{G} v_1$. But then by weak positive response $v_2 \preceq_{G} v_1$ in contradiction to condition 3.

Therefore, $P_G(v_1) = S_G(v_1) = \emptyset$. Thus, $G_1$ and $G_2$ satisfy the conditions of the weak union condition with regard to $G$. Therefore, $v \preceq_{F_G} v_1 \Rightarrow v \preceq_{F_G} v_2$, in contradiction to condition 1, because the edge set is the same and $|V| < |V_1|$. The Sixth Intl. Joint Conf. on Autonomous Agents and Multi-Agent Systems (AAMAS 07)
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Theorem 1. Assume correctness for a ranking system must not rely on the number of vertices in the graph, and moreover, that the minimal nontrivially ranked graph for a given ranking system must be connected.

8. THE ISOMORPHISM PROPERTY AND FURTHER RESEARCH

Most of the ranking systems we have seen up to now in the possibility proofs take advantage of the names of the vertices to determine the ranking. A natural requirement from a ranking system is that the names assigned to the vertices will not take part in determining the ranking. This is formalized by the isomorphism property.

Definition 13. A ranking system $F$ satisfies isomorphism if for every isomorphism function $\phi : V_1 \rightarrow V_2$, and two isomorphic graphs $G \in G_{V_1}$, $\phi(G) \in G_{V_2}$: $\phi(\phi(G)) = \phi(\phi(G))$.

It turns out that the ranking system $F_1$ from the possibility proof for weak incentive compatibility and weak monotonicity (Proposition 4) satisfies isomorphism as well, and thus there exists a weakly incentive compatible ranking system satisfying isomorphism and weak monotonicity. The existence of weakly incentive compatible ranking systems satisfying isomorphism in conjunction with either the weak union property or the weak positive response is an open question.

Another natural extension of this work, is to consider weaker notions of incentive compatibility, where agents may have beneficial deviations, but the amount or magnitude of such deviations is bounded. In a pending paper, we address
the quantification of such weaker notions of incentive compatibility.

9. REFERENCES


